(1) $\int \frac{3}{x^2}dx$

$=\int 3x^{-2}\,dx$

$=3\cdot(-1)\cdot x^{-1}+C$

$=-\frac{3}{x}+C$

(2) $\int \frac{3x-1}{x^2-2x-3}dx$

$\frac{3x-1}{x^2-2x-3}=\frac{3x-1}{(x-3)(x+1)}=\frac{a}{x-3}+\frac{b}{x+1}$

を満たす a,b を求める。

$$\frac{a}{x-3}+\frac{b}{x+1}=\frac{a(x+1)+b(x-3)}{(x-3)(x+1)}$$

$$=\frac{(a+b)x+(a-3b)}{(x-3)(x+1)}$$

なので,

$$\begin{cases} a+b=3 \\ a-3b=-1 \end{cases}$$

よって,

$a=2, b=1$

これより

$$\int \frac{3x-1}{x^2-2x-3}dx$$

$$=\int\left(\frac{2}{x-3}+\frac{1}{x+1}\right)dx$$

$$=2\log|x-3|+\log|x+1|+C$$

(3) $\int 2^{x+3}dx$

$=\frac{2^{x+3}}{log2}+C$

(4) $\int cos^2xdx$

$=\int \frac{1+\cos 2x}{2}dx$

$=\int(\frac{1}{2}+\frac{\cos 2x}{2})\,dx$

$=\frac{1}{2}x+\frac{1}{4}\sin 2x+C$

(5) $\int sin2xcos3xdx$

$=\int \frac{1}{2}(\sin 5x+\sin(-x))dx$

$=\int \frac{1}{2}(\sin 5x-\sin x)dx$

$=-\frac{1}{10}cos5x+cosx+C$

(6) $\int x(2x+1)^4dx$

$2x+1=t$つまり

$x=\frac{t-1}{2}$とおくと,

$\frac{dx}{dt}=\frac{1}{2}$より

$dx=\frac{1}{2}dt$

よって,

$\int x(2x+1)^4dx$

$=\int(\frac{t-1}{2}\cdot t^4\cdot\frac{1}{2})dt$

$=\int\left(\frac{1}{4}t^5-\frac{1}{2}t^4\right)dt$

$=\frac{1}{24}t^6-\frac{1}{10}t^5+C$

したがって,

$\int x(2x+1)^4dx$

$=\frac{1}{24}(2x+1)^6-\frac{1}{10}(2x+1)^5+C$

(7) $\int 3x^2e^{x^3}dx$

$u=x^3$とおくと,

$du=3x^2dx$

よって,

$$\int 3x^2e^{x^3}dx$$

$=\int e^u\cdot du$

$=e^u+C$

$=e^{x^3}+C$

(8) $\int tanxdx$

$=\int \frac{\sin x}{\cos x}dx$

$=\int \frac{-(cosx)'}{cosx}dx$

$=-\log|cosx|+C$

(9) $\int(x+1)sinxdx$

$= \int(x+1)(-cosx)'dx$

$=-(x+1)cosx+\int cosxdx$

$=-(x+1)cosx+sinx+C$

第２問 以下の定積分を計算せよ。

(1) $\int_1^4 2\sqrt{x}\,dx$

$=\int_1^4 2x^{\frac{1}{2}}dx$

$=\left[2\cdot\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$

$=\left(2\cdot\frac{2}{3}\cdot4^{\frac{3}{2}}\right)-\left(2\cdot\frac{2}{3}\cdot1^{\frac{3}{2}}\right)$

$=\frac{32}{3}-\frac{4}{3}$

$=\frac{28}{3}$

(2) $\int_1^3 e^{2x+1}\,dx$

$=\left[\frac{1}{2}e^{2x+1}\right]_1^3$

$=\left(\frac{1}{2}e^7\right)-\left(\frac{1}{2}e^3\right)$

$=\frac{1}{2}(e^7-e^3)$

(3) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} tan^2xdx$

$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{sin^2x}{cos^2x}\right)dx$

$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1-cos^2x}{cos^2x}\right)dx$

$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1}{cos^2x}-1\right)dx$

$=[\tan x-x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$=\left(\tan\frac{\pi}{3}-\frac{\pi}{3}\right)-\left(\tan\frac{\pi}{6}-\frac{\pi}{6}\right)$

$=\frac{2\sqrt{3}}{3}-\frac{\pi}{6}$

(4) $\int_e^{e^2} logxdx$

$=\int_e^{e^2}(x)'logxdx$

$= [xlogx]_e^{e^2}-\int_e^{e^2}x(logx)'dx$

$=(e^2loge^2)-(eloge)-\int_e^{e^2}dx$

$=2e^2-e-[x]_e^{e^2}$

$=2e^2-e-(e^2-e)$

$= e^2$

(5) $\int_0^{\pi}|\boldsymbol{sinx+cosx}|\boldsymbol{dx}$

$sinx+cosx \geqq 0$ を $0 \leqq x \leqq \pi$ の範囲で解くと,

$0 \leqq x \leqq \frac{3}{4}\pi$

すなわち、

$0 \leqq x \leqq \frac{3}{4}\pi$ で $sinx+cosx \geqq 0$,
$\frac{3}{4}\pi \leqq x \leqq \pi$ で $sinx+cosx \leqq 0$

よって,

$$\int_0^{\pi}|sinx+cosx|dx$$

$=\int_0^{\frac{3}{4}\pi}(sinx+cosx)dx+\int_{\frac{3}{4}\pi}^{\pi}-(sinx+cosx)dx$

$=[-cosx+sinx]_0^{\frac{3}{4}\pi}-[-cosx+sinx]_{\frac{3}{4}\pi}^{\pi}$

$=2\sqrt{2}-\frac{\sqrt{3}}{2}-\frac{1}{2}$

(6) $\int_0^{\sqrt{3}}\sqrt{\boldsymbol{9-x^2}}\,\boldsymbol{dx}$

$$x=3sin\theta$$

とおくと,

$$\frac{dx}{d\theta}=3cos\theta$$

$$dx=3cos\theta d\theta$$

$\int_0^{\sqrt{3}}\sqrt{9-x^2}\,dx$

$=\int_0^{\frac{\pi}{6}}\sqrt{9-9sin^2\theta}\,3cos\theta d\theta$

$=\int_0^{\frac{\pi}{6}}3|cos\theta|\,3cos\theta d\theta$

$0\leqq\theta\leqq\frac{\pi}{6}$ において,$0<cos\theta$ なので,

$$\int_0^{\frac{\pi}{6}}3|cos\theta|\,3cos\theta d\theta$$

$=9\int_0^{\frac{\pi}{6}}cos^2\theta\,d\theta$

$=9\int_0^{\frac{\pi}{6}}\frac{1+cos2\theta}{2}d\theta$

$=9\left[\frac{1}{2}\theta+\frac{1}{4}sin2\theta\right]_0^{\frac{\pi}{6}}$

$=\frac{3}{4}\pi+\frac{9}{8}\sqrt{3}$

(7) $\int_0^2 \frac{dx}{4+x^2}$

$$x = 2tan\theta$$

とおくと，

$$\frac{dx}{d\theta} = 2\frac{1}{cos^2\theta}$$

$$dx = 2\frac{1}{cos^2\theta}d\theta$$

$$\int_0^2 \frac{dx}{4+x^2} = \int_0^{\frac{\pi}{4}} \frac{1}{4+4tan^2\theta} \cdot 2\frac{1}{cos^2\theta}d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4(1+1tan^2\theta)} \cdot 2\frac{1}{cos^2\theta}d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4(1+1tan^2\theta)} \cdot 2\frac{1}{cos^2\theta}d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4\frac{1}{cos^2\theta}} \cdot 2\frac{1}{cos^2\theta}d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}d\theta$$

$$= \left[\frac{1}{2}\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}$$

(8) $\int_0^{\pi} (sinx + cosx)^2 dx$

$$= \int_0^{\pi} (sin^2x + 2sinxcosx + cos^2x)dx$$

$$= \int_0^{\pi} (1 + 2sinxcosx)dx$$

$$= \int_0^{\pi} (1 + cos2x)dx$$

$$= \left[x + \frac{1}{2}sin2x\right]_0^{\pi}$$

$$= \left(\pi + \frac{1}{2}sin2\pi\right) - \left(0 + \frac{1}{2}sin0\right)$$

$$= \pi + 2$$

第 3 問

**$t=\tan\frac{x}{2}$とおくとき,次の問いに答えよ。**

**(1)$\cos x$と$\sin x$を$t$を用いて表せ。**

**(2)$\frac{dt}{dx}$を$t$を用いて表せ。**

**(3)$\int\frac{dx}{3sinx+4cosx}$を求めよ。**

(1) $tan$の 2 倍角の公式から,

$$tanx=\frac{2tan\frac{x}{2}}{1-2tan^2\frac{x}{2}}$$

$$=\frac{2t}{1-t^2}$$

三角関数の相互関係の式より、

$$tan^2\frac{x}{2}+1=\frac{1}{cos^2\frac{x}{2}}$$

$$cos^2\frac{x}{2}=\frac{1}{tan^2\frac{x}{2}+1}$$

$$=\frac{1}{t^2+1}$$

2 倍角の公式から,

$$cosx=2cos^2\frac{x}{2}-1$$

$$=\frac{2}{t^2+1}-1$$

$$=\frac{1-t^2}{t^2+1}$$

また、三角比の相互関係から,

$$tanx=\frac{sinx}{cosx}$$

$$sinx=tanxcosx$$

$$=\frac{2t}{1-t^2}\cdot\frac{1-t^2}{t^2+1}$$

$$=\frac{2t}{t^2+1}$$

(2) $t=\tan\frac{x}{2}$

$$\frac{dt}{dx}=\frac{1}{2cos^2\frac{x}{2}}$$

$$=\frac{1}{2(\frac{1}{t^2+1})}$$

$$=\frac{t^2+1}{2}$$

(3) (1),(2)より,

$$\int\frac{dx}{3sinx+4cosx}$$

$$=\int\frac{1}{3\frac{2t}{t^2+1}+4\frac{1-t^2}{t^2+1}}\cdot\frac{2}{1+t^2}dt$$

$$=\int\frac{dt}{3t+2(1-t^2)}$$

$$=\int\frac{dt}{-2t^2+3t+2}$$

$$=\int\frac{dt}{(-2t-1)(t-2)}$$

ここで,

$$\frac{a}{t-2}-\frac{b}{-2t-1}=\frac{1}{(t-2)(-2t-1)}$$

を満たす$a,b$を求めると,

$$a=-\frac{1}{5},b=\frac{2}{5}$$

なので,

$$\int\frac{dt}{(-2t-1)(t-2)}$$

$$=-\frac{1}{5}\int\frac{dt}{t-2}-\frac{2}{5}\int\frac{dt}{2t+1}$$

$$=-\frac{1}{5}\log|t-2|-\frac{1}{5}\log|2t+1|+C$$

$$=-\frac{1}{5}\log\left|\frac{t-2}{2t+1}\right|+C$$

$$=\frac{1}{5}\log\left|\frac{2tan\frac{x}{2}+1}{tan\frac{x}{2}-2}\right|+C$$

第４問

I=$\int_0^{\frac{\pi}{2}} e^x \sin x\, dx$，J=$\int_0^{\frac{\pi}{2}} e^x \cos x dx$をそれぞれ

求めよ。

I=$\int_0^{\frac{\pi}{2}} e^x \sin x\, dx$

$=\int_0^{\frac{\pi}{2}} (e^x)' sinx\, dx$

$=[e^x sinx]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x cosxdx$

I=$e^{\frac{\pi}{2}} - J$　　・・・①

また、

J=$\int_0^{\frac{\pi}{2}} e^x \cos x dx$

$=\int_0^{\frac{\pi}{2}} (e^x)' cosx\, x$

$=[e^x cosx]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x sinxdx$

$=-1$ +I　　・・・②

①＋②より、

2J=$e^{\frac{\pi}{2}} - 1$

J=$\frac{e^{\frac{\pi}{2}}-1}{2}$

①−②より、

2I=$e^{\frac{\pi}{2}} + 1$

J=$\frac{e^{\frac{\pi}{2}}+1}{2}$